Math 121 3.1 Graphing Using the First Derivative

Objectives

1) Find the critical numbers of a function (also called critical values)

2) Use the first derivative to identify open intervals where the function is increasing or decreasing.

3) Use the first derivative test to determine if a critical number is the location of
   - a relative maximum
   - a relative minimum
   - neither a relative maximum nor relative minimum

4) Use first derivative information to sketch the graph of a function

5) Use first derivative information plus vertical and horizontal asymptotes to sketch the graph of a rational function.

A <u>relative maximum</u> is a y-coordinate on the graph which is greater than its nearby neighbors. Plural: maxima

A <u>relative minimum</u> is a y-coordinate on the graph which is less than its nearby neighbors. Plural: minimum

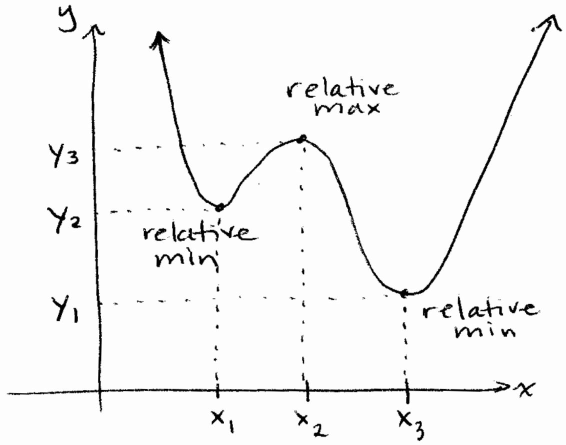
A <u>relative extremum</u> or <u>relative extreme value</u> is either a relative max or a relative min. Plural: extrema.

Note: A relative max does not have to be the biggest y-value on the entire graph — that would be the absolute maximum.

Similarly, a relative minimum may not be the smallest y-value on the entire graph.

When graphing, relative extrema are key points to find and include on the graph!

ex.

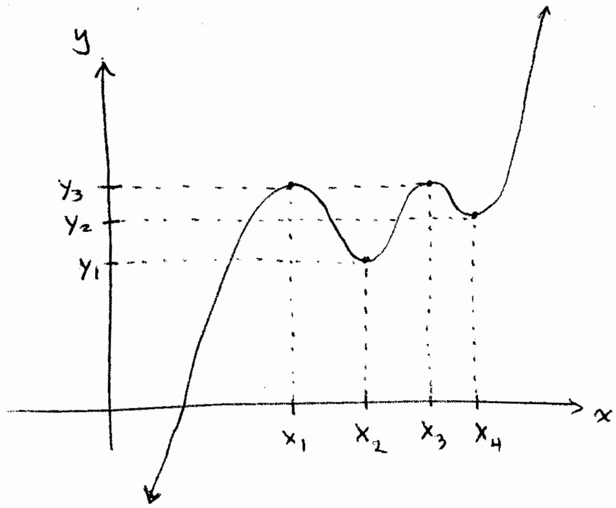


y_3 is a relative max occurring at x_2

y_2 is a relative min occurring at x_1

y_1 is a relative min occurring at x_3

ex.

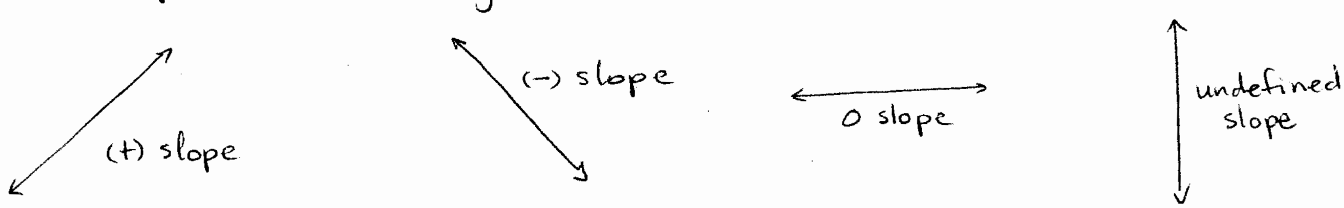


y_3 is a relative max occurring at x_1 and x_3.

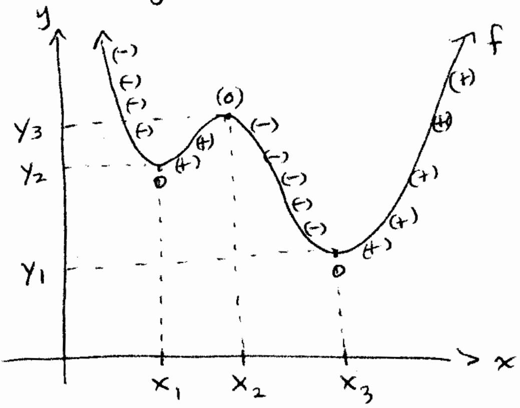
y_2 is a relative min occurring at x_4

y_1 is a relative min occurring at x_2

Recall: The first derivative of $f(x)$ is $f'(x)$ and $f'(x)$ tells us the slope of the tangent line to $f(x)$ at x.

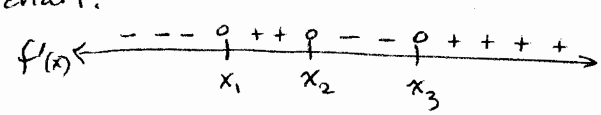


① Make a sign chart for $f'(x)$ given this graph of $f(x)$.



If we write (+) near the graph to indicate positive tangent line slopes, (−) for negative, 0 for zero slope and ϕ for undefined, we can learn information about $f'(x)$.

sign chart:



Caution: We will be making sign charts for $f''(x)$ also, so make it a habit, please, to always label your sign charts

Regarding intervals of increase or decrease

- Different textbooks or sources treat endpoints and asymptotes differently.

Ex: ① Some sources use closed intervals

increasing $(-\infty, 3]$

decreasing $[3, \infty)$

(even though this creates some confusion about what is happening at $x=3$)

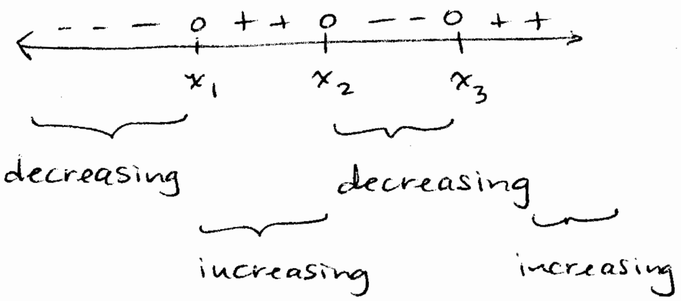
Our book says "open intervals"

Ex. ⑦ Some sources disregard asymptotes when writing intervals of increase, decrease.

decreasing $(-\infty, \infty)$

(even though this glosses over the fact that $f(3)$ is undefined and f is nondifferentiable at $x=3$ ($f'(3)$ also undefined.)

(2) Using the sign chart from the previous example, write the open intervals of increase or decrease.



intervals of decrease
$(-\infty, x_1) \cup (x_2, x_3)$

intervals of increase
$(x_1, x_2) \cup (x_3, \infty)$

Note: Some sources sometimes have closed intervals of increase/decrease. Our book uses only open intervals.

From a sign chart:

(1) f' changes from $\underset{\text{increasing}}{(+)}$ to $\underset{\varnothing \text{ or}}{0}$ to $\underset{\text{decreasing}}{(-)}$ $\Rightarrow$ relative max

(2) f' changes from $\underset{\text{decreasing}}{(-)}$ to $\underset{\varnothing \text{ or}}{0}$ to $\underset{\text{increasing}}{(+)}$ $\Rightarrow$ relative min

(3) f' is $\underset{\text{increasing}}{(+)}$ to $\underset{\varnothing \text{ or}}{0}$ to $\underset{\text{increasing}}{(+)}$ $\Rightarrow$ neither max nor min.

(4) f' is $\underset{\text{decreasing}}{(-)}$ to $\underset{\varnothing \text{ or}}{0}$ to $\underset{\text{decreasing}}{(-)}$ $\Rightarrow$ neither max nor min

Using a sign chart to analyze the change of sign of $f'(x)$ is called the first derivative test for relative extreme values.

A value x is called a critical number (or critical value) of a function f if

a) $(x, f(x))$ is a point on the graph

b) $\underbrace{f'(x) = 0}_{m_{tan} = 0}$ or $\underbrace{f'(x) \text{ is undefined}}_{m_{tan} \text{ undefined}}$.

$m_{tan} = 0$ $\downarrow$ horizontal tangent

m_{tan} undefined $\downarrow$ vertical tangent

③ Use algebra to answer.

$f(x) = -x^2 + 6x - 5$

a) What shape is the graph?

b) Does the graph have a relative max, relative min, neither, or both?

c) Find the relative extrema.

d) Sketch the graph.

a) $f(x) = -x^2 + 6x - 5$ is a quadratic function, so its graph is a $\boxed{\text{parabola.}}$

b) Because the leading coefficient $a = -1$ is negative, the parabola opens downward, so the function has a $\boxed{\text{relative max. only.}}$

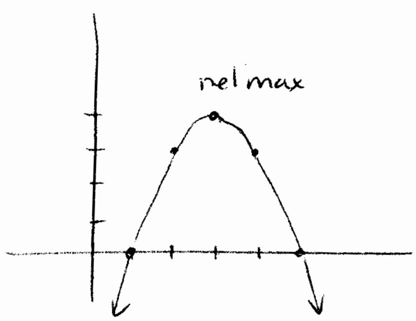
c) The max occurs at the vertex

$$x = \frac{-b}{2a} = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3$$

$$f(3) = -3^2 + 6(3) - 5 = 4$$

$\boxed{\text{max value } 4 \text{ at } x = 3}$

d)



④ Confirm our work in ③ by using calculus.

$f'(x) = -2x + 6$

$f'(x) = 0$ means $-2x + 6 = 0$

$$-2x = -6$$

$$x = 3$$

$f(3) = $ y-coordinate $= 4$

sign chart f': (+) (+) (+) 0 (−) (−) (−), with 0 at $x = 3$

test $x = 0$: $f'(0) = 6$ (+)

test $x = 4$: $f'(4) = -2(4) + 6 = -2$ (−)

shape of $f(x)$: ↗ rel max ↘

Horizontal Asymptotes: (Limits at Infinity (2.1))

- Find highest degree term in denominator of rational function (call this degree n)
- Multiply all terms, top and bottom, by $\frac{1}{x^n}$.
- Simplify
- Use $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ whenever n is positive.

Vertical Asymptotes

- factor numerator and denominator to simplify
- if any factor cancels, the associated value of x is a hole (and not a vertical asymptote)
- set denominator = 0 and isolate x

CAUTION:
Vertical and Horizontal Asymptotes are features of Rational Functions (fractions)

$$\frac{p(x)}{q(x)}$$

but not of polynomial functions

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

Summary

A value of $x=c$ is called a critical number (or critical value)
if a) $f(c)$ is defined $\Rightarrow$ there is a point on the graph at $(c, f(c))$

AND b) $f'(c)=0$ or $f'(c)$ undefined

First Derivative Test to determine if $x=c$, a critical number, is the location of a relative max, relative min, or neither.

- Plot $x=c$ on a number line for all critical numbers.
- Determine the sign of the first derivative between the critical numbers
- If the sign changes (+) to (−) left to right of $x=c$, then it is the location of ↗ ↘ a relative maximum, $f(c)$.
- If the sign changes (−) to (+) left to right of $x=c$, then it is the location of ↘ ↗ a relative minimum, $f(c)$.
- If the signs don't change, it is neither a max nor a min.

When graphing:

- Find the domain of the function.
- Find the first derivative of the function, and all the critical numbers.
- Make a sign chart for the first derivative
- Identify the intervals of increase or decrease.
- Identify the relative extrema and their y-coordinates (by substituting critical numbers into original function.)
- Plot all the relative extrema — choose axes that make this possible.
- Plot all vertical and horizontal asymptotes.
- Sketch the increasing and decreasing directions between relative extrema and asymptotes.

For each example

a) Find the critical numbers

b) Make a signchart for $f'(x)$.

c) Write the intervals of increase or decrease.

d) Identify the relative extrema using the first derivative test.

e) Sketch the graph, including any asymptotes.

5 $f(x) = (x-3)^{4/7} + 4$

a) $f'(x) = \underbrace{\frac{4}{7}(x-3)^{-3/7} \cdot 1}_{\text{chain rule}} + \underbrace{0}_{\text{constant}}$

$$f'(x) = \frac{4}{7}(x-3)^{-3/7} = \frac{4}{7(x-3)^{3/7}}$$

$f'(x) = 0$ means $\frac{4}{7(x-3)^{3/7}} = 0$ — mult both sides by LCD

$$\cancel{7(x-3)^{3/7}} \cdot \frac{4}{\cancel{7(x-3)^{3/7}}} = 0 \cdot 7 \cdot (x-3)^{3/7}$$

$$4 \neq 0$$

There are no locations x where the slope of the tangent line is 0.
NO HORIZONTAL TANGENTS.

$f'(x)$ undefined means $\div 0$.
Set denominator of $f'(x)$ to 0.

$$7(x-3)^{3/7} = 0$$

$$(x-3)^{3/7} = \frac{0}{7} = 0$$

$$(\sqrt[7]{x-3})^3 = 0$$

$$\sqrt[7]{x-3} = 0^{1/3} = 0$$

$$x-3 = 0^7 = 0$$

$$\boxed{x = 3 \text{ is a critical number}}$$

b) $f'(x)$: $\underbrace{(-)\ (-)\ (-)}_{\text{test } x=0}$ ϕ at 3 $\underbrace{(+)\ (+)\ (+)}_{\text{test } x=4}$

$f'(0) = \frac{4}{7(0-3)^{3/7}} \approx -.36$

$f'(4) = \frac{4}{7(4-3)^{3/7}} = \frac{4}{7}$ (+)

c) $\boxed{\begin{array}{l}\text{increasing } (3, \infty) \\ \text{decreasing } (-\infty, 3)\end{array}}$

d) Because f' changes from $(-)$ to $(+)$ at $x = 3$, there is a relative minimum when $(3, f(3))$.

rel min

$= (3, 4)$

$f(3) = (3-3)^{4/7} + 4$

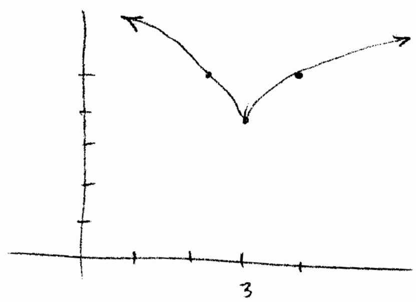
$= 0^{4/7} + 4$

$= 0 + 4$

$= 4$

$\boxed{\text{minimum value 4 occurs at } x = 3}$

e)



vertical tangent at $x = 3$ creates a cusp.

$f(2) = 5$

$f(4) = 5$

(6) $f(x) = \dfrac{3x+4}{x-2}$

a) $f'(x) = \dfrac{(x-2)\cdot\frac{d}{dx}(3x+4) - (3x+4)\cdot\frac{d}{dx}(x-2)}{(x-2)^2}$ Quotient Rule

$= \dfrac{(x-2)\cdot 3 - (3x+4)\cdot 1}{(x-2)^2}$

$= \dfrac{3x - 6 - 3x - 4}{(x-2)^2}$

$f'(x) = \dfrac{-10}{(x-2)^2}$

$f'(x) = 0$ $\dfrac{-10}{(x-2)^2} = 0$

$-10 \neq 0$ no horizontal tangent lines

$f'(x)$ undefined $x - 2 = 0$

$x = 2$ vertical asymptote

$f(2) = \dfrac{3(2)+4}{2-2} = \dfrac{10}{0} = $ undefined.

Since $f(2)$ is undefined,

$\boxed{\text{there are no critical numbers}}$

b) f' $(-)$ $(-)$ $(-)$ $(-)$

test $x = 0$ $\dfrac{-10}{(0-2)^2} = \dfrac{-10}{4} = $ neg

c) $\boxed{\text{decreasing } (-\infty, 2), (2, \infty) \quad \text{or} \quad (-\infty, \infty)}$

d) $\boxed{\text{no relative extrema}}$ because f' does not change sign

e) Find vertical asymptotes: denom $= 0$

$\underline{\underline{x = 2}}$

Find horizontal asymptotes:

$$\lim_{x\to\infty} \frac{3x+4}{x-2}$$

mult all by $\frac{1}{x}$ (using highest power from denominator)

$$= \lim_{x\to\infty} \frac{\frac{3x}{x}+\frac{4}{x}}{\frac{x}{x}-\frac{2}{x}}$$

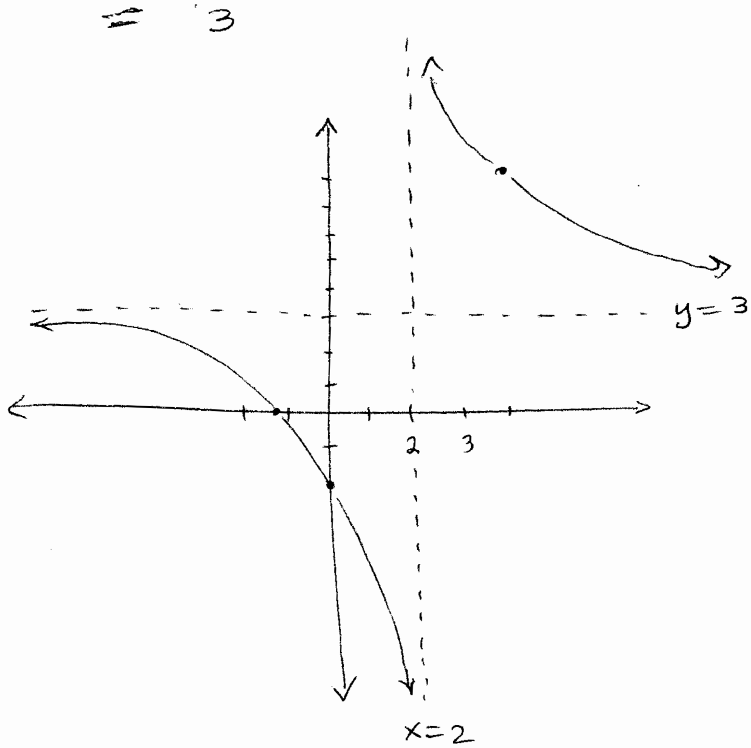
$$= \lim_{x\to\infty} \frac{3+\frac{4}{x}}{1-\frac{2}{x}}$$

$$= \frac{3+0}{1-2}$$

$$= 3$$

As $x \to \pm\infty$, y approaches 3.

Horizontal asymptote $\underline{\underline{y=3}}$



- plot asymptotes
- a few anchor points
- connect with decreasing curves

x-int
set $y=0$

$0 = \frac{3x+4}{x-2}$

$0 = 3x+4$

$-\frac{4}{3} = x$

y-int
set $x=0$

$\frac{3(0)+4}{0-2}$

$= \frac{4}{-2}$

$y = -2$

$x=3$

$f(3) = \frac{3(3)+4}{3-2} = 13$ too high

$f(4) = \frac{16}{2} = 8$

7. Find asymptotes and sketch graph of $f(x) = \dfrac{3x^2}{x^2-4}$

vertical asymptotes: denom $= 0$

$$x^2 - 4 = 0$$
$$x = \pm\sqrt{4}$$
$$\boxed{x=2} \text{ and } \boxed{x=-2}$$

horizontal asymptotes: $\frac{1}{x^n}$, denom $x^2 \to \frac{1}{x^2}$ is used

$$\lim_{x\to\infty} \frac{3x^2 \cdot \frac{1}{x^2}}{\frac{1}{x^2}\cdot x^2 - 4\cdot\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3}{1-\frac{4}{x^2}}$$

$$= \frac{3}{1-0}$$

$$= 3$$

horizontal asymptote $\boxed{y=3}$

$$f'(x) = \frac{(x^2-4)\cdot 6x - 3x^2(2x)}{(x^2-4)^2}$$

$$= \frac{6x^3 - 24x - 6x^3}{(x^2-4)^2}$$

$$= \frac{-24x}{(x^2-4)^2}$$

$f'(x) = 0 \qquad -24x = 0$

$x = 0.$

$f(0)$ defined? $\quad f(0) = \dfrac{3(0)^2}{0^2-4} = \dfrac{0}{-4} = 0$ ✓

This is a critical number

$f'(x)$ undefined $\qquad x^2 - 4 = 0$

$x = \pm 2$

$f(2)$ undefined

$f(-2)$ undefined

These are not critical numbers.

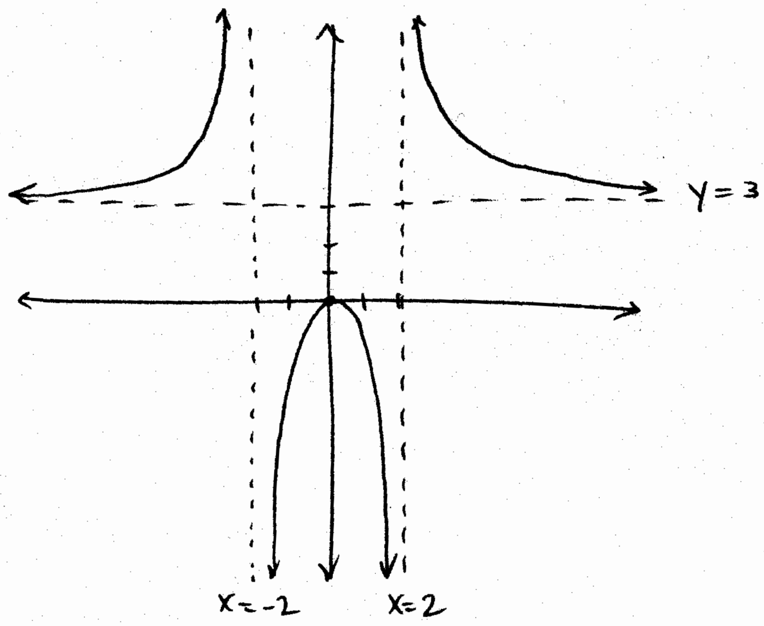
f': (+) (+) ∅ (+) 0 (−) ∅ (−) (−)

test $x=-3$	-2	test $x=-1$	0	test $x=1$	2	test $x=3$

$f'(-3) = \frac{72}{25} \qquad f'(-1) = \frac{8}{3} \qquad f'(1) = \frac{-8}{3} \qquad f'(3) = \frac{-72}{25}$

at $x=0$ there is a relative maximum,

$f(0) = \dfrac{3(0)^2}{0^2-4} = 0 \qquad (0,0).$



x-int: set y=0
y-int: set x=0 $\Big\}$ (0,0)

increasing $(-\infty,-2) \cup (-2,0)$
decreasing $(0,2) \cup (2,\infty)$